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Dynamic behavior of Moving distributed Masses of Orthotropic Rectangular Plate with Clamped-Clamped Boundary conditions Resting on a Constant Elastic Bi-Parametric Foundation

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Abstract**—** *This work will investigate the behaviors of Moving distributed masses of orthotropic rectangular plates resting on bi-parametric elastic foundation with clamped-clamped end conditions. The governing equation is a fourth order partial differential equation with variable and singular co-efficients. The solutions to the problem will be obtained by transforming the partial differential equation for the problem to a set of coupled second order ordinary differential equations using the technique of Shadnam et al[18] which shall then be simplified using modified asymptotic method of Struble. The closed form solution will be analyzed, resonance condition shall be obtained and the result will be presented in plotted curves for both cases of moving distributed mass and moving distributed force.*

Keywords**— *Bi-parametric foundation,orthotropic, foundation modulus, critical speed, resonance, modified frequency.***

I. INTRODUCTION

A plate is a flat structural element for which the thickness is small compared with the surface dimensions or a plate is a structural element which is thin and flat. By "thin", it means that the plate's transverse dimension, or thickness, is small in comparison with the length and width dimensions. That is, the plate thickness is small compared to the other dimensions. A mathematical expression of this idea is:

$$\frac{T}{L} \ll 1 \tag{1.1}$$

where $'T'$ represents the plate's thickness, and 'L' represents the length or width dimension. The thickness of a plate is usually constant but may be variable and is measured normal to the middle surface of the plate. Plates subjected to in-plane loading can be solved mostly by using two-dimensional plane stress theory. On the other hand,plate theory is concerned mainly with lateral loading. There are whole lot of differences between plane stress and plate theory. One of the differences is that in the plate theory,the stress components are allowed to vary through the thickness of the plate,so that there would exist bending moments.

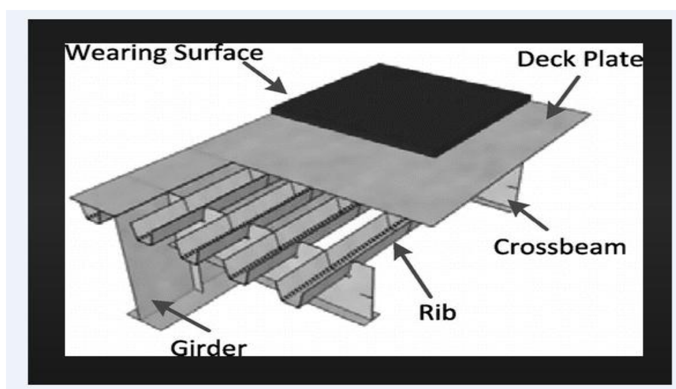

Fig.1.1: The Skeletal Structure of Orthotropic Plate

The vast majority of the world's bridges with record long spans have utilized orthotropic steel deck systems as their superstructures.These types of decks have been used

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extensively in Europe, Asia and South America. Approximately 6,000 orthotropic steel deck bridges exist in a world of about six million bridges. With the growing trend quicker construction practices with an overall longer bridge life. The other leading benefits of this bridge decking system are the minimization of dead load in the design and the rapid construction that will lessen the impact on traffic.

Tracing the plate theory to its roots, one travels back to the American revolution time. During this revolutionary period, several scientists and engineers performed numerous researches on plates. Euler [1] performed a free vibration analysis of plate problems and indicated the first impetus to a mathematical statement of plate problems. Chladni [2], a German physicist, performed experiments on horizontal plates to quantify their vibratory modes. He spattered sand on the plates, struck them with a hammer, and observed there were regular patterns formed along the nodal lines. Bernoulli [3] then attempted to theoretically justify the experimental results of Chladni using the previously developed Euler-Bernoulli bending beam theory, but his results were unable to capture the full dynamics. The French mathematician Germain [4] developed a plate differential equation that lacked a twisting term but one of the reviewers of her works, Lagrange [5], corrected Germain’s results. Thus, he was the first person to present the general plate equation as stated by Ventsel and Krauthemmer [6]. Cauchy [7] and Poisson [8] developed the problem of plate bending using general theory of elasticity. Poison successfully expanded, the Germain-Lagrange plate equation to the solution of a plate uder static loading. In this solution, the plate flexural rigidity was set to be a constant term D, as showed in Ventsel and Krauthammer [6]. Navier [9] considered the plate thickness in the general plate equation as a function of rigidity D. Kirchhoff [10] shed more light on thin plate theory based on some assumptions that are now referred to as "Kirchhoff’s hypothesis". These hypotheses are fundamental assumptions in the development of linear, elastic, small-deflection theory for the bending of thin plates. It is based on Kirchhoff hypothesis that straight lines normal to the undeformed mid-plane remain straight and normal to the deformed mid-plane. In accordance with the kinematic assumptions made in the classical plate theory, all the transverse shear and transverse normal strains are zero. In the works of these great scientists, the flexural rigidity was considered to be constant, D. That is to say, they all treated plates as homogeneous and isotropic materials, that their material properties remain unchanged in all directions but in reality plates are orthotropic.

In the 1920’s, American engineers began using steel plate riveted to steel beams for large movable bridges. The purpose was to minimize the dead load of the lift span. In 1938, the American Institute of Steel Construction (AISC) began publishing reports on the steel-deck system. AISC called this the "battledeck floor" because it felt the steel deck had the strength of a battleship. The orthotropic deck was a result of the "battledeck floor". This floor consisted of a steel deck plate, supported by longitudinal (normally I-beam) stringers. In their turn, these stringers were supported by cross beams. Following World War II, German engineers developed the modern orthotropic bridge design as a response to material shortages during the post-war period. The orthotropic deck reduced the weight of continuous beams considerably and permitted spans and slenderness ratios unknown until then. Since the 1950s, steel highway bridges have been constructed with a steel deck structure. This structure is called orthotropic deck. These decks were constructed with a deck plate supported by stiffeners of various shapes and by cross-beams and main girders. The deck plate acts as the top flange of the deck girders.

Since 1965, a new generation of orthotropic decks has been used, with cold-formed trapezoidal stringers, so-called troughs. This trough permitted cross-beam spacing. An orthotropic deck or orthotropic bridge is one whose deck typically comprises a structural steel deck plate stiffened either longitudinally or transversely, or in both directions. This allows the deck both to directly bear vehicular loads and to contribute to the bridge structure’s overall load-bearing behaviour. The orthotropic deck may be integral with or supported on a grid of deck framing members such as floor beams and girders. Decks with different stiffnesses in longitudinal and transverse directions are called ’orthotropic’. If the stiffnesses are similar in the two directions, then the deck is called ’isotropic’. The steel deck-plate-and-ribs system may be idealized for analytical purposes as an orthogonal-anisotropic plate, hence the abbreviated designation ’orthotropic.’ In vibration analysis of plates, the effect of elastic foundation must be considered when plates are mounted on elastic springs such as pavement of roads, footing of buildings and bases of machines.Extensive understanding of vibration behavior of orthotropic plates is important in many engineering fields. Therefore, the vibration analysis of plates resting on elastic foundation has been the subject of many researches. Ugural [11] and Okafor and

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Oguaghamba [12] applied Ritz and Galerkin's methods to solve isotropic and orthotropic plate problems. Their results are very close to those exact solutions. Some researchers applied superposition method for analysis of vibrations of plates having classical boundary conditions for instance Ohya et al [13] for deformable plates.Awodola and Omolofe [14] investigated the response of concentrated moving masses of elastically supported rectangular plates resting on Winlker elastic foundation by the method of variable separable. Agarana, Gbadeyan and Ajayi [15] solved an isotropic elastic damped rectangular Mindlin plate resting on Pasternak foundation by using numerical method. Are, Idowu and Gbadeyan [16] dealt with a damped simply supported orthotropic rectangular plates resting on elastic Winkler foundation by applying method of separation of variables to reduce the problem to second order coupled differential equation and then applied numerical methods to obtain the solution. Gbadeyan and Dada [17] improved on their previous works by considering the dynamic response of a Mindlin elastic rectangular plate subjected to distributed moving load, but neglected the effect of damping.

The orthotropic materials are of a great importance and interest in fields of the modern industrial technologies applications due to their hardness,lightness and super elasticity.The problem shall be solved analytically, which is not found in the work of the early researchers by using the technique of Shadnam et al[18]. Barlie [19] carried out the thermo-mechanical morphological study of CFRP in different environmental conditions. Guo [20, 21] examined the fatigue lif of reinforced concrete beams strengthened with CFRP under beading load.Hosseini [22] investigated the roadway metallic bridge using non-prestressed bonded and prestressed unbonded CFRP plate. In the same view, Ju [23] and Li [24] examined the fatigue life of reinforced and non-prestressed bridge desk under variable amplitude load and different adhesives. Liang [25] studied the reliability analysis of bond behavior of CFRP concrete under wet-dry cycles. Xin Yuan [2] also investigated the fatigue performance and life prediction of CFRP plate.

II. GOVERNING EQUATION

The dynamic transverse displacement $V(x,y,t)$ of orthotropic rectangular plates when it is resting on a bi-parametric elastic foundation and traversed by distributed mass M_r moving with constant velocity c_r along a straight line parallel to the x-axis issuing from point y=s on the y-axis with flexural rigidities D_x and D_y is governed by the fourth order partial differential equation given as

$$\begin{aligned}&D_x\frac{\partial^4}{\partial x^4}W(x,y,t)+2B\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)+D_y\frac{\partial^4}{\partial y^4}W(x,y,t)+\mu\frac{\partial^2}{\partial t^2}W(x,y,t)-\rho hR_0\\&[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]+K_0W(x,y,t)-G_0[\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]\\&-\sum_{r=1}^{N}[M_rgH(x-ct)H(y-s)-M_r(\frac{\partial^2}{\partial t^2}W(x,y,t)+2c_r\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t)\\&)H(x-c_rt)H(y-s)W(x,y,t)]=0 \qquad (2.1)\end{aligned}$$

where D_x and D_y are the flexural rigidities of the plate along x and y axes respectively.

$$D_x=\frac{E_xh^3}{12(1-\nu_x\nu_y)},\quad D_y=\frac{E_yh^3}{12(1-\nu_x\nu_y)},\quad B=D_xD_y+\frac{G_oh^3}{6}$$

E_x and E_y are the Young's moduli along x and y axes respectively, G_o is the rigidity modulus, ν_x and ν_y are Poisson's ratios for the material such that $E_x\nu_y = E_y\nu_x$, ρ is the mass density per unit volume of the plate, h is the plate thickness, t is the time, x and y are the spatial coordinates in x and y directions respectively, R_o is the rotatory inertia correction factor, K_o is the foundation constant and g is the acceleration due to gravity, H(.) is the Heaviside function.

Rewriting equation (2.1), one obtains

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$$\mu\frac{\partial^2}{\partial t^2}W(x,y,t)+\mu\omega_n^2W(x,y,t)=\rho hR_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]-2B$$
$$\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-D_x\frac{\partial^4}{\partial x^4}W(x,y,t)-D_y\frac{\partial^4}{\partial y^4}W(x,y,t)-K_0W(x,y,t)+\mu\omega_n^2W(x,y,t)$$
$$+G_0[\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]+\sum_{r=1}^{N}[M_rgH(x-c_rt)H(y-s)-M_r(\frac{\partial^2}{\partial t^2}W(x,y,t)$$
$$+2c\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-c_rt)H(y-s)W(x,y,t)] \quad (2.2)$$

which can be expressed further as

$$\frac{\partial^2}{\partial t^2}W(x,y,t)+\omega_n^2W(x,y,t)=R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]-\frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}$$
$$W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t)+[\omega_n^2-\frac{K_0}{\mu}]W(x,y,t)+\frac{G_0}{\mu}[\frac{\partial^2}{\partial x^2}$$
$$W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]+\sum_{r=1}^{N}[\frac{M_rg}{\mu}H(x-c_rt)H(y-s)-\frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t)+2c \quad (2.3)$$
$$\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-c_rt)H(y-s)W(x,y,t)]$$

where ω_n^2 is the natural frequencies, $n=1,2,3,\ldots$ The initial conditions, without any loss of generality, is taken as

$$W(x,y,t)=0=\frac{\partial}{\partial t}W(x,y,t) \quad (2.4)$$

III. ANALYTICAL APPROXIMATE SOLUTION

In order to solve equation (2.3), one applies technique of Shadnam et al[19] which requires that the deflection of the plates be in series form as

$$W(x,y,t)=\sum_{n=1}^{N}\Phi_n(x,y)Q_n(t) \quad (3.1)$$

where $\Phi_n(x,y)=\Phi_{ni}(x)\Phi_{nj}(y)$ and

$$\Phi_{ni}(x)=\sin\frac{\nu_{ni}}{L_x}x+A_{ni}\cos\frac{\nu_{ni}}{L_x}x+B_{ni}\sinh\frac{\nu_{ni}}{L_x}x+C_{ni}\cosh\frac{\nu_{ni}}{L_x}x$$

$$\Phi_{ni}(y)=\sin\frac{\nu_{nj}}{L_y}y+A_{nj}\cos\frac{\nu_{nj}}{L_y}y+B_{nj}\sinh\frac{\nu_{nj}}{L_y}y+C_{nj}\cosh\frac{\nu_{nj}}{L_y}y \quad (3.2)$$

The right hand side of equation (2.3) written in the form of series takes the form

$$\sum_{n=1}^{\infty}R_0[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)]-\frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)$$
$$-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t+[\omega_n^2-\frac{K_0}{\mu}]W(x,y,t)+\frac{G_0}{\mu}[\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)]+ \quad (3.3)$$
$$\sum_{r=1}^{N}[\frac{M_rg\mu H(x-c_rt)}{H}(y-s)-\frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t)+2c_r\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}$$
$$W(x,y,t))H(x-c_rt)H(y-s)W(x,y,t)]=\sum_{n=1}^{N}\Phi_n(x,y)\sigma_n(t)$$

On multiplying both sides of equation (3.3) by $\Phi_m(x,y)$, integrating on area A of the plate and considering the orthogonality of $\Phi_m(x,y)$, one obtains

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$$\sigma_n(t) = \frac{1}{\Delta}\sum_{n=1}^{\infty}\int_A [R_0(\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t) + \frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)) - \frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t) - \frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t) - \frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t + (\omega_n^2 - \frac{K_0}{\mu})W(x,y,t) + \frac{G_0}{\mu}(\frac{\partial^2}{\partial x^2}W(x,y,t) + \frac{\partial^2}{\partial y^2}W(x,y,t)) + \sum_{r=1}^{N}[\frac{M_r g}{\mu}H(x-c_r t)H(y-s) - \frac{M_r}{\mu}(\frac{\partial^2}{\partial t^2}W(x,y,t) + 2c_r\frac{\partial^2}{\partial x\partial t}W(x,y,t) + c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t))H(x-c_r t)H(y-s)W(x,y,t)]\Phi_m(x,y)dA \quad (3.4)$$

and zero when $n \neq m$

where

$$\theta^* = \int_A \Phi_n^2(x,y)dA \quad (3.5)$$

Making use of equation (3.1) and taking into account equations (2.3) and (3.4), equation (3.3) can be written as

$$\Phi_n(x,y)[\ddot{Q}_n(t) + \omega_n^2 Q_n(t)] = \frac{\Phi_n(x,y)}{\Delta}\sum_{q=1}^{\infty}\int_A [R_0(\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)\ddot{Q}_q(t) + \frac{\partial^2\Phi_q(x,y)}{\partial y^2}\Phi_m(x,y)\ddot{Q}_q(t)) - \frac{2B}{\mu}\frac{\partial^2\Phi_q(x,y)}{\partial x^2\partial y^2}\Phi_m(x,y)Q_q(t) - \frac{D_x}{\mu}\frac{\partial^4\Phi_q(x,y)}{\partial x^4}\Phi_m(x,y)Q_q(t) - \frac{D_y}{\mu}\frac{\partial^4\Phi_q(x,y)}{\partial y^4}\Phi_m(x,y)Q_q(t) + (\omega_n^2 - \frac{K_0}{\mu})\Phi_q(x,y)\Phi_m(x,y)Q_q(t) + \frac{G_0}{\mu}(\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t) + \frac{\partial^2\Phi_q(x,y)}{\partial y^2}\Phi_m(x,y)Q_q(t)) + \sum_{r=1}^{N}(\frac{M_r g\mu\Phi_m(x,y)H(x-c_r t)}{H}(y-s) - \frac{M_r}{\mu}(\Psi_q(x,y)\Phi_m(x,y)\ddot{Q}_q(t) + 2c_r\frac{\partial\Phi_q(x,y)}{\partial x}\Phi_m(x,y)\dot{Q}_q(t) + c_r^2\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t)) H(x-c_r t)H(y-s))]dA \quad (3.6)$$

On further simplification of equation (3.6), one obtains

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = \frac{1}{\Delta}\sum_{q=1}^{\infty}\int_A [R_0(\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)\ddot{Q}_q(t) + \frac{\partial^2\Phi_q(x,y)}{\partial y^2}\Phi_m(x,y)\ddot{Q}_q(t)) - \frac{2B}{\mu}\frac{\partial^2\Phi_q(x,y)}{\partial x^2\partial y^2}\Phi_m(x,y)Q_q(t) - \frac{D_x}{\mu}\frac{\partial^4\Phi_q(x,y)}{\partial x^4}\Phi_m(x,y)Q_q(t) - \frac{D_y}{\mu}\frac{\partial^4\Phi_q(x,y)}{\partial y^4}\Phi_m(x,y) Q_q(t) + (\omega_n^2 - \frac{K_0}{\mu})\Phi_q(x,y)\Phi_m(x,y)Q_q(t) + \frac{G_0}{\mu}(\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t) + \frac{\partial^2\Phi_q(x,y)}{\partial y^2}\Phi_m(x,y)Q_q(t)) + \sum_{r=1}^{N}(\frac{M_r g}{\mu}\Phi_m(x,y)H(x-c_r t)H(y-s) - \frac{M_r}{\mu}(\Phi_q(x,y)\Phi_m(x,y)\ddot{Q}_q(t) + 2c_r\frac{\partial\Phi_q(x,y)}{\partial x}\Phi_m(x,y)\dot{Q}_q(t) + c_r^2\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t))H(x-c_r t)H(y-s))]dA \quad (3.7)$$

The system in equation (3.7) is a set of coupled ordinary differential equations

Using the Fourier series representation, the Heaviside functions take the form

$$H(x-c_r t) = \frac{1}{4} + \frac{1}{\pi}\sum_{r=1}^{N}\frac{\sin(2n+1)\pi(x-c_r t)}{2n+1}, 0 < x < 1 \quad (3.8)$$

$$H(y-s) = \frac{1}{4} + \frac{1}{\pi}\sum_{r=1}^{N}\frac{\sin(2n+1)\pi(y-s)}{2n+1}, 0 < y < 1 \quad (3.9)$$

On putting equations (3.8) and (3.9) into equation (3.7) and simplifying one obtains

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$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) - \frac{1}{\Delta}\sum_{q=1}^{\infty} [R_0 T_0 \ddot{Q}_q(t) - \frac{2B}{\mu} T_1 Q_q(t) - \frac{D_x}{\mu} T_2 Q_q(t) - \frac{D_y}{\mu} T_3 Q_q(t) + (\omega_n^2 - \frac{K_0}{\mu}) T_4 Q_q(t) + \frac{G_0}{\mu} T_5 Q_q(t) - \sum_{r=1}^{N} \frac{M_r}{\mu}((T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_1^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_2^* \frac{\sin(2j+1)\pi c_r t}{2j+1})(\sum_{k=1}^{\infty} E_3^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi} (\sum_{j=1}^{\infty} E_5^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_6^* \frac{\sin(2j+1)\pi c_r t}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_7^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^* \frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_q(t) + 2c_r t(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_9^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_{10}^* \frac{\sin(2j+1)\pi c_r t}{2j+1})(\sum_{k=1}^{\infty} E_{11}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{12}^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_{14}^* \frac{\sin(2j+1)\pi c_r t}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^* \frac{\sin(2k+1)\pi s}{2k+1}))\dot{Q}_q(t) + c_r^2(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_{18}^* \frac{\sin(2j+1)\pi c_r t}{2j+1})(\sum_{k=1}^{\infty} E_{19}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^* \frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_{22}^* \frac{\sin(2j+1)\pi c_r t}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^* \frac{\sin(2k+1)\pi s}{2k+1})) Q_q(t))] = \sum_{q=1}^{\infty} \sum_{r=1}^{N} \frac{M_r g}{\mu\Delta} \Phi_m(ct)\Phi_m(s) \tag{3.10}$$

which is the transformed equation governing the problem of an orthotropic rectangular plate resting on bi-parametric elastic foundation.

where

$$T_0 = \int_A [\frac{\partial^2}{\partial x^2}\Phi_q(x,y)\Phi_m(x,y) + \frac{\partial^2}{\partial y^2}\Phi_q(x,y)\Phi_m(x,y)]dA \tag{3.11}$$

$$T_1 = \int_A \frac{\partial^2}{\partial x^2}[\frac{\partial^2}{\partial x^2}\Phi_q(x,y)]\Phi_m(x,y)dA \tag{3.12}$$

$$T_2 = \int_A \frac{\partial^4}{\partial x^4}[\Phi_q(x,y)]\Phi_m(x,y)dA \tag{3.13}$$

$$T_3 = \int_A \frac{\partial^4}{\partial y^4}[\Phi_q(x,y)]\Phi_m(x,y)dA \tag{3.14}$$

$$T_4 = \int_A \Phi_q(x,y)\Phi_m(x,y)dA \tag{3.15}$$

$$T_5 = \int_A [\frac{\partial^2}{\partial x^2}\Phi_q(x,y) + \frac{\partial^2}{\partial y^2}\Phi_q(x,y)]\Phi_m(x,y)dA \tag{3.16}$$

$$T_6 = \frac{1}{16}\int_A \Phi_q(x,y)\Phi_m(x,y)dA \tag{3.17}$$

$$E_1^* = \int_A \Phi_q(x,y)\Phi_m(x,y)\sin(2j+1)\pi x dA \tag{3.18}$$

$$E_2^* = \int_A \Phi_q(x,y)\Phi_m(x,y)\cos(2j+1)\pi x dA \tag{3.19}$$

$$E_3^* = \int_A \Phi_q(x,y)\Phi_m(x,y)\sin(2k+1)\pi y dA \tag{3.20}$$

$$E_4^* = \int_A \Phi_q(x,y)\Phi_m(x,y)\cos(2k+1)\pi y dA \tag{3.21}$$

$$E_5^* = E_1^*, \quad E_6^* = E_2^*, \quad E_7^* = E_3^*, \quad E_8^* = E_4^* \tag{3.22}$$

$$T_7 = \frac{1}{16}\int_A \frac{\partial}{\partial x}\Phi_q(x,y)\Phi_m(x,y)dA \tag{3.23}$$

$$E_9^* = \int_A \frac{\partial}{\partial x}(\Phi_q(x,y))\Phi_m(x,y)\sin(2j+1)\pi x dA \tag{3.24}$$

$$E_{10}^* = \int_A \frac{\partial}{\partial x}(\Phi_q(x,y))\Phi_m(x,y)\cos(2j+1)\pi x dA \tag{3.25}$$

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$$E_{11}^{*}=\int_{A}\frac{\partial}{\partial x}(\Phi_{q}(x,y))\Phi_{m}(x,y)\sin(2k+1)\pi y dA \quad (3.26)$$

$$E_{12}^{*}=\int_{A}\frac{\partial}{\partial x}\Phi_{q}(x,y)\Phi_{m}(x,y)\cos(2k+1)\pi y dA \quad (3.27)$$

$$E_{13}^{*}=E_{9}^{*},\quad E_{14}^{*}=E_{10}^{*},\quad E_{15}^{*}=E_{11}^{*},\quad E_{16}^{*}=E_{12}^{*} \quad (3.28)$$

$$T_{8}=\frac{1}{16}\int_{A}\frac{\partial^{2}}{\partial x^{2}}(\Phi_{q}(x,y))\Phi_{m}(x,y)dA \quad (3.29)$$

$$E_{17}^{*}=\int_{A}\frac{\partial^{2}}{\partial x^{2}}(\Phi_{q}(x,y))\Phi_{m}(x,y)\sin(2j+1)\pi x dA \quad (3.30)$$

$$E_{18}^{*}=\int_{A}\frac{\partial^{2}}{\partial x^{2}}(\Phi_{q}(x,y))\Phi_{m}(x,y)\cos(2j+1)\pi x dA \quad (3.31)$$

$$E_{19}^{*}=\int_{A}\Phi_{q}(x,y)\Phi_{m}(x,y)\sin(2k+1)\pi y dA \quad (3.32)$$

$$E_{20}^{*}=\int_{A}\frac{\partial^{2}}{\partial x^{2}}(\Phi_{q}(x,y))\Phi_{m}(x,y)\cos(2k+1)\pi y dA \quad (3.33)$$

$$E_{21}^{*}=E_{17}^{*},\quad E_{22}^{*}=E_{18}^{*},\quad E_{23}^{*}=E_{19}^{*},\quad E_{24}^{*}=E_{20}^{*} \quad (3.34)$$

$\Phi_{m}(x,y)$ is assumed to be the products of functions $\Phi_{pm}(x)\Phi_{bm}(y)$ which are the beam functions in the directions of x and y axes respectively. That is

$$\Phi_{m}(x,y)=\Phi_{pm}(x)\Phi_{bm}(y) \quad (3.35)$$

where

$$\begin{gathered}\Phi_{pm}(x)=\sin\lambda_{pm}x+A_{pm}\cos\lambda_{pm}x+B_{pm}\sinh\lambda_{pm}x+C_{pm}\cosh\lambda_{pm}x\\ \Phi_{bm}(y)=\sin\lambda_{bm}+A_{bm}\cos\lambda_{bm}y+B_{bm}\sinh\lambda_{bm}y+C_{bm}\cosh\lambda_{bm}y\end{gathered} \quad (3.36)$$

where A_{pm}, B_{pm}, C_{pm}, A_{bm}, B_{bm} and C_{bm} are constants determined by the boundary conditions. And Φ_{pm} and Φ_{bm} are called the mode frequencies

where

$$\lambda_{pm}=\frac{\xi_{pm}}{L_{x}},\qquad \lambda_{bm}=\frac{\xi_{bm}}{L_{y}} \quad (3.37)$$

Considering a unit mass, equation (3.10) can be rewritten as

$$\ddot{Q}_{n}(t)+\omega_{n}^{2}Q_{n}(t)-\frac{1}{\Delta}\sum_{q=1}^{\infty}[R_{0}T_{0}\ddot{Q}_{q}(t)-\frac{2B}{\mu}T_{1}Q_{q}(t)-\frac{D_{x}}{\mu}T_{2}Q_{q}(t)-\frac{D_{y}}{\mu}T_{3}Q_{q}(t)$$
$$+(\omega_{n}^{2}-\frac{K_{0}}{\mu}T_{4})Q_{q}(t)+\frac{G_{0}}{\mu}T_{5}Q_{q}(t)-\varpi\varphi((T_{6}+\frac{1}{\pi^{2}}(\sum_{j=1}^{\infty}E_{1}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-$$
$$\sum_{j=1}^{\infty}E_{2}^{*}\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty}E_{3}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{4}^{*}\frac{\sin(2k+1)\pi s}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_{5}^{*}$$
$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{6}^{*}\frac{\sin(2j+1)\pi ct}{2j+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_{7}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{8}^{*}$$
$$\frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_{q}(t)+2c(T_{7}+\frac{1}{\pi^{2}}(\sum_{j=1}^{\infty}E_{9}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{10}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}$$

$$)(\sum_{k=1}^{\infty}E_{11}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{j=1}^{\infty}E_{12}^{*}\frac{\sin(2k+1)\pi s}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_{13}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-$$
$$\sum_{j=1}^{\infty}E_{14}^{*}\frac{\sin(2j+1)\pi ct}{2j+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_{15}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{16}^{*}\frac{\sin(2k+1)\pi s}{2k+1}))\dot{Q}_{q}(t)$$

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$$+c^2(T_8+\frac{1}{\pi^2}(\sum_{j=1}^{\infty}E_{17}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty}E_{19}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{20}^*\frac{\sin(2k+1)\pi s}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_{21}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{22}^*\frac{\sin(2j+1)\pi ct}{2j+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}))Q_q(t))] = \sum_{q=1}^{\infty}\frac{Mg}{\mu\Delta}\Phi_m(ct)\Phi_m(s) \quad (3.38)$$

equation (3.38) is the fundamental equation of the problem. where

$$\varpi=\frac{M}{\mu\varphi},\quad \varphi=L_xL_y \quad (3.39)$$

$$\Phi_m(ct)=\sin\alpha_m(t)+A_m\cos\alpha_m(t)+B_m\sinh\alpha_m(t)+C_m\cosh\alpha_m(t) \quad (3.40)$$

$$\Phi_m(s)=\sin\lambda_m+A_m\cos\lambda_m+B_m\sinh\lambda_m+C_m\cosh\lambda_m \quad (3.41)$$

$$\alpha_m=\frac{\Gamma_m c}{L_x},\ \lambda_m=\frac{\Gamma_m s}{L_y} \quad (3.42)$$

3.1 Orthotropic Rectangular Plate Traversed by a Moving Force

In moving force, we account for only the load being transferred to the structure. In this case, the inertia effect is negligible. Setting $\phi=0$ in the fundamental equation (3.38), one obtains

$$\ddot{Q}_n(t)+(1-\frac{T_4}{\mu\Delta})\omega_n^2Q_n(t)-\frac{1}{\mu\Delta}[\mu R_0T_0\ddot{Q}_n(t)-2BT_1Q_n(t)-D_xT_2Q_n(t)-D_yT_3Q_n(t)-K_0T_4Q_n(t)+G_0T_5Q_n(t)+\sum_{q=1,q\neq n}^{\infty}(\mu R_0T_0\ddot{Q}_q(t)-2BT_1Q_q(t)-D_xT_2Q_q(t)-D_yT_3Q_q(t)+(\mu\omega_q^2-K_0T_4)Q_q(t)+G_0T_5Q_q(t))]=\frac{Mg}{\mu\Delta}\Phi_m(ct)\Phi_m(s) \quad (3.43)$$

which can further be simplified as

$$\ddot{Q}_n(t)+\xi_n^2Q_n(t)-\gamma[\mu R_0T_0\ddot{Q}_n(t)-2BT_1Q_n(t)-D_xT_2Q_n(t)-D_yT_3Q_n(t)-K_0T_4Q_n(t)+G_0T_5Q_n(t)+\sum_{q=1,q\neq n}^{\infty}(\mu R_0T_0\ddot{Q}_q(t)-2BT_1Q_q(t)-D_xT_2Q_q(t)-D_yT_3Q_q(t)+(\mu\omega_q^2-K_0T_4)Q_q(t)+G_0T_5Q_q(t))]=\gamma Mg\Phi_m(ct)\Phi_m(s) \quad (3.44)$$

where $\xi_n^2=(1-\frac{T_4}{\mu\Delta})\omega_n^2$

Expanding and re-arranging equation (3.44), one obtains

$$[1-\gamma\mu R_0T_0]\ddot{Q}_n(t)+(\xi_n^2-\gamma J_6)Q_n(t)-\gamma\sum_{q=1,q\neq n}^{\infty}(\mu R_0T_0\ddot{Q}_q(t)-2BT_1Q_q(t)-D_xT_2Q_q(t)-D_yT_3Q_q(t)+(\mu\omega_q^2-K_0T_4)Q_q(t)+G_0T_5Q_q(t))=\gamma Mg\Phi_m(ct)\Phi_m(s) \quad (3.45)$$

Simplifying further, one obtains

$$\ddot{Q}_n(t)+\frac{(\xi_n^2-\gamma J_6)}{[1-\gamma\mu R_0T_0]}Q_n(t)+\frac{\gamma}{[1-\gamma\mu R_0T_0]}\sum_{q=1,q\neq n}^{\infty}(\mu R_0T_0\ddot{Q}_q(t)-2BT_1Q_q(t)-D_xT_2Q_q(t)-D_yT_3Q_q(t)+(\mu\omega_q^2-K_0T_4)Q_q(t)+G_0T_5Q_q(t))=\frac{\gamma Mg}{[1-\gamma\mu R_0T_0]}\Phi_m(ct)\Phi_m(s) \quad (3.46)$$

where

$$\gamma=\frac{1}{\mu\Delta},\quad J_6=-2BT_1-D_xT_2-D_yT_3-K_0T_4+G_0T_5 \quad (3.47)$$

For any arbitrary ratio γ, defined as

$\gamma^*=\frac{\gamma}{1+\gamma}$, one obtains

$$\gamma=\frac{\gamma^*}{1-\gamma^*}=\gamma^*+o(\gamma^{*2})+\ldots$$

For only $o(\gamma^*)$, one obtains

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$$\gamma = \gamma^*$$

On application of binomial expansion,

$$\frac{1}{1-\gamma^*\mu R_0 T_0} = 1 + \gamma^*\mu R_0 T_0 + o(\gamma^{*2}) + \ldots \tag{3.48}$$

On putting equation (3.48) into equation (3.46), one obtains

$$\ddot{Q}_n(t) + (\xi_n^2 - \gamma^* J_6)(1 + \gamma^*\mu R_0 T_0 + o(\gamma^{*2}) + \ldots)Q_n(t) + \gamma^*(1 + \gamma^*\mu R_0 T_0 + o(\gamma^{*2}) + \ldots)\sum_{q=1,q\neq n}^{\infty}(\mu R_0 T_0 \ddot{Q}_q(t) - 2BT_1 Q_q(t) - D_x T_2 Q_q(t) - D_y T_3 Q_q(t) + (\mu\omega_q^2 - K_0 T_4)Q_q(t) + G_0 T_5 Q_q(t)) = \gamma^* a M g(1 + \gamma^*\mu R_0 T_0 + o(\gamma^{*2}) + \ldots)\Phi_m(ct)\Phi_m(s) \tag{3.49}$$

Retaining only $o(\gamma^*)$, equation (3.49) becomes

$$\ddot{Q}_n(t) + [\xi_n^2(1 + \gamma^*\mu R_0 T_0) - \gamma^* J_6]Q_n(t) + \gamma^*\sum_{q=1,q\neq n}^{\infty}(\mu R_0 T_0 \ddot{Q}_q(t) - 2BT_1 Q_q(t) - D_x T_2 Q_q(t) - D_y T_3 Q_q(t) + (\mu\omega_q^2 - K_0 T_4)Q_q(t) + G_0 T_5 Q_q(t)) = \gamma^* M g \Phi_m(ct)\Phi_m(s) \tag{3.50}$$

which is simplified further as

$$\ddot{Q}_n(t) + \xi_n^2 Q_n(t) + \gamma^*\sum_{q=1,q\neq n}^{\infty}(\mu R_0 T_0 \ddot{Q}_q(t) - 2BT_1 Q_q(t) - D_x T_2 Q_q(t) - D_y T_3 Q_q(t)\ (\mu\omega_q^2 - K_0 T_4)Q_q(t) + G_0 T_5 Q_q(t)) = \gamma^* M g \Phi_m(ct)\Phi_m(s) \tag{3.51}$$

where

$$J_7 = [\xi_n^2(1 + \gamma^*\mu R_0 T_0) - \gamma^* J_6] \tag{3.52}$$

Using Struble's technique, one obtains

$$\xi_{nn} = \xi_n - \left(\frac{\xi_n^2 - J_7}{2\xi_n}\right) \tag{3.53}$$

which is the modified frequency for moving force problem.

Using equation (3.53), the homogeneous part of equation (3.51) can be written as

$$\ddot{Q}_n(t) + \xi_{nn}^2 Q_n(t) = 0 \tag{3.54}$$

Hence, the entire equation (3.51) gives

$$\ddot{Q}_n(t) + \xi_{nn}^2 Q_n(t) = \gamma^* M g \Phi_m(ct)\Phi_m(s) \tag{3.55}$$

On solving equation (3.55), one obtains

$$Q_n(t) = \frac{Mg\gamma^*\Phi_m(s)}{\xi_{nn}(\alpha_m^4 - \xi_{nn}^4)}[(\alpha_m^2 + \xi_{nn}^2)(\alpha_m \sin\xi_{nn}t - \xi_{nn}\sin\alpha_m t) - A_m \xi_{nn}(\alpha_m^2 + \xi_{nn}^2)(\cos\alpha_m t - \cos\xi_{nn}t) - B_m(\alpha_m^2 - \xi_{nn}^2)(\alpha_m \sin\xi_{nn}t - \xi_{nn}\sinh\alpha_m t) + C_m \xi_{nn}(\alpha_m^2 - \xi_{nn}^2)(\cosh\alpha_m t - \cos\xi_{nn}t)] \tag{3.56}$$

which on inversion yields

$$W(x,y,t) = \sum_{pm=1}^{\infty}\sum_{qm=1}^{\infty}\frac{Mg\gamma^*\Phi_m(s)}{\xi_{nn}(\alpha_m^4 - \xi_{nn}^4)}[(\alpha_m^2 + \xi_{nn}^2)(\alpha_m \sin\xi_{nn}t - \xi_{nn}\sin\alpha_m t) - A_m \xi_{nn}(\alpha_m^2 + \xi_{nn}^2)(\cos\alpha_m t - \cos\xi_{nn}t) - B_m(\alpha_m^2 - \xi_{nn}^2)(\alpha_m \sin\xi_{nn}t - \xi_{nn}\sinh\alpha_m t) + C_m \xi_{nn}(\alpha_m^2 - \xi_{nn}^2)(\cosh\alpha_m t - \cos\xi_{nn}t)](\sin\frac{\xi_{pm}}{L_x}x + A_{pm}\cos\frac{\xi_{pm}}{L_x}x + B_{pm}\sinh\frac{\xi_{pm}}{L_x}x + C_{pm}\cosh\frac{\xi_{pm}}{L_x}x)(\sin\frac{\xi_{qm}}{L_y}y + A_{qm}\cos\frac{\xi_{qm}}{L_y}y + B_{qm}\sinh\frac{\xi_{qm}}{L_y}y + C_{qm}\cosh\frac{\xi_{qm}}{L_y}y) \tag{3.57}$$

which is the transverse displacement response to a moving force.

3.2 Orthtropic Rectangular Plate Traversed by a Moving Mass

In moving mass problem, the moving load is assumed rigid, and the weight and as well as inertia forces are transferred to the moving load. That is the inertia effect is not negligible. Thus $\varpi \neq 0$ and so it is required to solve the entire equation (3.38).To

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solve the equation,one employs analytical approximate method. This method is known as an approximate analytical method of Struble. The homogeneous part of equation (3.38) shall be replaced by a free system operator defined by the modified frequency ξ_{nn}. Thus, the entire equation becomes

$$\ddot{Q}_n(t) + \xi_{nn}^2 Q_n(t) + \varpi\varphi^* \sum_{q=1}^{\infty} [(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_1^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^* \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_3^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_5^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_6^* \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_7^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^* \frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_q(t) + 2c(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_9^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^* \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_{11}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{j=1}^{\infty} E_{12}^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{14}^* \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^* \frac{\sin(2k+1)\pi s}{2k+1}))\dot{Q}_q(t) + c^2(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^* \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_{19}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{22}^* \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^* \frac{\sin(2k+1)\pi s}{2k+1}))Q_q(t)] = \sum_{q=1}^{\infty} \frac{Mg}{\mu\Delta}\Phi_m(ct)\Phi_m(s) \quad (3.58)$$

where $\varphi^* = \frac{1}{\mu\varepsilon^*}$

On expanding equation (3.58), one obtains

$$\ddot{Q}_n(t) + \xi_{nn}^2 Q_n(t) + \varpi\varphi^*[(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_1^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^* \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_3^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^* \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_5^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_6^* \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_7^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^* \frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_n(t) + 2c(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_9^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^* \frac{\sin(2j+1)\pi ct}{2j+1})$$

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$$-\sum_{j=1}^{\infty} E_{14}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$))\dot{Q}_n(t) + c^2(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})$$
$$(\sum_{k=1}^{\infty} E_{19}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$-\sum_{j=1}^{\infty} E_{22}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$))Q_n(t)] + \varpi\varphi^{*} \sum_{q=1, q\neq n}^{\infty} [(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_1^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^{*}$$
$$\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_3^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_5^{*}$$
$$\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_6^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_7^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^{*}$$
$$\frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_q(t) + 2c(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_9^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}$$
$$)(\sum_{k=1}^{\infty} E_{11}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{j=1}^{\infty} E_{12}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$-\sum_{j=1}^{\infty} E_{14}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$))\dot{Q}_q(t) + c^2(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})$$

$$(\sum_{k=1}^{\infty} E_{19}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$-\sum_{j=1}^{\infty} E_{22}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*} \frac{\sin(2k+1)\pi s}{2k+1} \quad (3.59)$$
$$))Q_q(t)] = Mg\varphi\Phi_m(ct)\Phi_m(s)$$

On rearranging and simplifying equation (3.59),one obtains

$$(1 + \varpi\varphi^{*}(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_1^{*} \frac{\cos(2j+1)\,\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^{*} \frac{\sin(2j+1)\,\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_3^{*}$$
$$\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_5^{*} \frac{\cos(2j+1)\pi ct}{2j+1} -$$
$$\sum_{j=1}^{\infty} E_6^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_7^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^{*} \frac{\sin(2k+1)\pi s}{2k+1})$$
$$))\ddot{Q}_n(t) + 2c\varpi\varphi^{*}(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_9^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}$$

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$$)(\sum_{k=1}^{\infty} E_{11}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{j=1}^{\infty} E_{12}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$- \sum_{j=1}^{\infty} E_{14}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$))\dot{Q}_n(t) + (\xi_{nn}^2 + \varpi\varphi^* c^2 (T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})$$
$$(\sum_{k=1}^{\infty} E_{19}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$- \sum_{j=1}^{\infty} E_{22}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$)))Q_n(t) + \varpi\varphi^* \sum_{q=1, q\neq n}^{\infty} [(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{1}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{2}^{*}$$
$$\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_{3}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{4}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{5}^{*}$$
$$\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{6}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{7}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{8}^{*}$$

$$\frac{\sin(2k+1)\pi s}{2k+1}))\ddot{Q}_q(t) + 2c(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{9}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}$$
$$)(\sum_{k=1}^{\infty} E_{11}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{j=1}^{\infty} E_{12}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$- \sum_{j=1}^{\infty} E_{14}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{15}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*} \frac{\sin(2k+1)\pi s}{2k+1}$$
$$))\dot{Q}_q(t) + c^2(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})$$

$$(\sum_{k=1}^{\infty} E_{19}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*} \frac{\sin(2k+1)\pi s}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^{*} \frac{\cos(2j+1)\pi ct}{2j+1}$$
$$- \sum_{j=1}^{\infty} E_{22}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + \frac{1}{4\pi}(\sum_{k=1}^{\infty} E_{23}^{*} \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*} \frac{\sin(2k+1)\pi s}{2k+1} \qquad (3.60)$$
$$))Q_q(t)] = Mg\varphi\Phi_m(ct)\Phi_m(s)$$

On further simplifications and re-arrangement, one obtains

$$\ddot{Q}_n(t) + 2c\varpi\varphi^*(T_7 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{9}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_{11}^{*}$$
$$\frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{12}^{*} \frac{\sin(2k+1)\pi ct}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{13}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{14}^{*}$$
$$\frac{\sin(2j+1)\pi ct}{2j+1}) + (\sum_{k=1}^{\infty} E_{15}^{*} \frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*} \frac{\sin(2k+1)\pi ct}{2k+1}))\dot{Q}_n(t) +$$
$$(\xi_{nn}^2(1 - \varpi\varphi^*(T_6 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{1}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{2}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})$$
$$(\sum_{k=1}^{\infty} E_{3}^{*} \frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{4}^{*} \frac{\sin(2k+1)\pi ct}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{5}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} -$$
$$\sum_{j=1}^{\infty} E_{6}^{*} \frac{\sin(2j+1)\pi ct}{2j+1}) + (\sum_{k=1}^{\infty} E_{7}^{*} \frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{8}^{*} \frac{\sin(2k+1)\pi ct}{2k+1})))) +$$
$$c^2\varpi\varphi^*(T_8 + \frac{1}{\pi^2}(\sum_{j=1}^{\infty} E_{17}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*} \frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty} E_{19}^{*}$$
$$\frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*} \frac{\sin(2k+1)\pi ct}{2k+1}) + \frac{1}{4\pi}(\sum_{j=1}^{\infty} E_{21}^{*} \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{22}^{*}$$
$$\frac{\sin(2j+1)\pi ct}{2j+1}) + (\sum_{k=1}^{\infty} E_{23}^{*} \frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*} \frac{\sin(2k+1)\pi ct}{2k+1}))Q_n(t)$$

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$$+\varpi\varphi^*\sum_{q=1,q\neq n}^{\infty}[(T_6+\frac{1}{\pi^2}(\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_2^*\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi ct}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_5^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi ct}{2j+1})+(\sum_{k=1}^{\infty}E_7^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_8^*\frac{\sin(2k+1)\pi ct}{2k+1}))\ddot{Q}_q(t)+$$
$$2c(T_7+\frac{1}{\pi^2}(\sum_{j=1}^{\infty}E_9^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{10}^*\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty}E_{11}^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_{12}^*\frac{\sin(2k+1)\pi ct}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_{13}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{14}^*\frac{\sin(2j+1)\pi ct}{2j+1})+(\sum_{k=1}^{\infty}E_{15}^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_{16}^*\frac{\sin(2k+1)\pi ct}{2k+1}))\dot{Q}_q(t)+c^2(T_8+\frac{1}{\pi^2}(\sum_{j=1}^{\infty}E_{17}^*$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1})(\sum_{k=1}^{\infty}E_{19}^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_{20}^*\frac{\sin(2k+1)\pi ct}{2k+1})+\frac{1}{4\pi}(\sum_{j=1}^{\infty}E_{21}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{22}^*\frac{\sin(2j+1)\pi ct}{2j+1})+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_{23}^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_{24}^*\frac{\sin(2k+1)\pi ct}{2k+1}))Q_q(t)]=Mg\varphi\Phi_m(ct)\Phi_m(s) \quad (3.61)$$

Applying modified asymptotic method of Struble, the solution to equation (3.61) becomes

$$\ddot{Q}_n(t)+\vartheta_n^2Q_n(t)=0 \quad (3.62)$$

Hence, the entire equation becomes

$$\ddot{Q}_n(t)+\vartheta_n^2Q_n(t)=Mg\varphi\Phi_m(ct)\Phi_m(s) \quad (3.63)$$

where

$$\vartheta_n=\xi_{nn}-\frac{1}{2\xi_{nn}}(\xi_{nn}^2\varpi\varphi^*(T_6+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_7^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_8^*\frac{\sin(2k+1)\pi s}{2k+1}))-c^2\varpi\varphi^*(T_8+\frac{1}{4\pi}(\sum_{k=1}^{\infty}E_{23}^*\frac{\cos(2k+1)\pi ct}{2k+1}-\sum_{k=1}^{\infty}E_{24}^*\frac{\sin(2k+1)\pi ct}{2k+1}))) \quad (3.64)$$

which is the modified frequency representing the frequency of the free system.

Rewriting equation (3.64), one obtains

$$\ddot{Q}_n(t)+\vartheta_n^2Q_n(t)=Mg\varphi\Phi_m(s)[\sin\alpha_m(t)+A_m\cos\alpha_m t+B_m\sinh\alpha_m t+C_m\cosh\alpha_m t] \quad (3.65)$$

Following the procedures applied to solve equation (3.51), one obtains

$$Q_n(t)=\frac{Mg\varphi\Phi_m(s)}{\vartheta_n(\alpha_m^4-\vartheta_n^4)}[(\alpha_m^2+\vartheta_n^2)(\alpha_m\sin\vartheta_n t-\vartheta_n\sin\alpha_m t)-A_m\vartheta_n(\alpha_m^2+\vartheta_n^2)(\cos\alpha_m t-\cos\vartheta_n t)-B_m(\alpha_m^2-\vartheta_n^2)(\alpha_m\sin\vartheta_n t-\vartheta_n\sinh\alpha_m t)+C_m\vartheta_n(\alpha_m^2-\vartheta_n^2)(\cosh\alpha_m t-\cos\vartheta_n t)] \quad (3.66)$$

which on inversion yields

$$W(x,y,t)=\sum_{pm=1}^{\infty}\sum_{qm=1}^{\infty}\frac{Mg\varphi\Phi_m(s)}{\vartheta_n(\alpha_m^4-\vartheta_n^4)}[(\alpha_m^2+\vartheta_n^2)(\alpha_m\sin\vartheta_n t-\vartheta_n\sin\alpha_m t)-A_m\vartheta_n(\alpha_m^2+\vartheta_n^2)(\cos\alpha_m t-\cos\vartheta_n t)-B_m(\alpha_m^2-\vartheta_n^2)(\alpha_m\sin\vartheta_n t-\vartheta_n\sinh\alpha_m t)+C_m\vartheta_n(\alpha_m^2-\vartheta_n^2)(\cosh\alpha_m t-\cos\vartheta_n t)](\sin\frac{\xi_{pm}}{L_x}x+A_{pm}\cos\frac{\xi_{pm}}{L_x}x+B_{pm}\sinh\frac{\xi_{pm}}{L_x}x+C_{pm}\cosh\frac{\xi_{pm}}{L_x}x)(\sin\frac{\xi_{qm}}{L_y}y+A_{qm}\cos\frac{\xi_{qm}}{L_y}y+B_{qm}\sinh\frac{\xi_{qm}}{L_y}y+C_{qm}\cosh\frac{\xi_{qm}}{L_y}y) \quad (3.67)$$

which is the transverse displacement response to a moving mass of a rectangular plate.

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IV. ILLUSTRATIVE EXAMPLES

4.1 Orthotropic Rectangular Plate Clamped at All Edges

For an orthotropic plate clamped at all its edges, the boundary conditions are given by

$$V(0,y,t)=0, \quad V(L_x,y,t)=0 \tag{4.1}$$

$$V(x,0,t)=0, \quad V(x,L_y,t)=0 \tag{4.2}$$

$$\frac{\partial V(0,y,t)}{\partial x}=0, \quad \frac{\partial V(L_x,y,t)}{\partial x}=0 \tag{4.3}$$

$$\frac{\partial V(x,0,t)}{\partial y}=0, \quad \frac{\partial V(x,L_y,t)}{\partial y}=0 \tag{4.4}$$

Thus,for the normal modes

$$\xi_{pm}(0)=0, \quad \xi_{pm}(L_x)=0 \tag{4.5}$$

$$\xi_{bm}(0)=0, \quad \xi_{bm}(L_y)=0 \tag{4.6}$$

$$\frac{\partial \xi_{pm}(0)}{\partial x}=0, \quad \frac{\partial \xi_{pm}(L_x)}{\partial x}=0 \tag{4.7}$$

$$\frac{\partial \xi_{bm}(0)}{\partial y}=0, \quad \frac{\partial \xi_{bm}(L_y)}{\partial y}=0 \tag{4.8}$$

For simplicity, our initial conditions are of the form

$$V(x,y,0)=0=\frac{\partial V(x,y,o)}{\partial t} \tag{4.9}$$

Using the boundary conditions in equations (4.1) to (4.4) and the initial conditions given by equation (4.9), it can be shown that

$$A_{pm}=\frac{\sinh\xi_{pm}-\sin\xi_{pm}}{\cos\xi_{pm}-\cosh\xi_{pm}}=\frac{\cos\xi_{pm}-\cosh\xi_{pm}}{\sin\xi_{pm}+\sinh\xi_{pm}} \tag{4.10}$$

$$A_{bm}=\frac{\sinh\xi_{bm}-\sin\xi_{bm}}{\cos\xi_{bm}-\cosh\xi_{bm}}=\frac{\cos\xi_{bm}-\cosh\xi_{bm}}{\sin\xi_{bm}+\sinh\xi_{bm}} \tag{4.11}$$

In the same vein, we have

$$A_m=\frac{\sinh\xi_m-\sin\xi_m}{\cos\xi_m-\cosh\xi_m}=\frac{\cos\xi_m-\cosh\xi_m}{\sin\xi_m+\sinh\xi_m} \tag{4.12}$$

$$B_{pm}=-1, \quad B_{bm}=-1, \quad \Rightarrow B_m=-1 \tag{4.13}$$

$$C_{pm}=-A_{pm}, \quad C_{bm}=-A_{bm}, \quad \Rightarrow C_m=-A_m \tag{4.14}$$

and from equation (4.12) , one obtains

$$\cos\xi_m\cosh\xi_m=1 \tag{4.15}$$

which is termed the frequency equation for the dynamical problem, such that

$$\xi_1=4.73004, \quad \xi_2=7.85320, \quad \xi_3=10.9951 \tag{4.16}$$

On using equations (4.12), (4.13), (4.14), (4.15) and (4.16) in equations (3.57) and (3.67), one obtains the displacement response to a moving force and a moving mass of clamped orthotropic rectangular plate resting on bi-parametric elastic foundation respectively.

V. DISCUSION OF THE ANALYTICAL SOLUTIONS

For this undamped system, it is desirable to examine the phenomenon of resonance. From equation (3.57), it is explicitly shown that

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the clamped orthotropic rectangular plate resting on constant elastic foundation and traverse by moving distributed force with uniform speed reaches a state of resonance whenever

$$\alpha_m = \xi_{nn} \tag{5.1}$$

while equation (3.67) shows that the same clamped orthotropic rectangular plate resting on constant elastic foundation and traverse by moving distributed force with uniform speed reaches a state of resonance when

$$\alpha_m = \vartheta_n \tag{5.2}$$

where

$$\begin{aligned}\vartheta_n = \xi_{nn} - \frac{1}{2\xi_{nn}}\Big(\xi_{nn}^2\varpi\varphi^*\Big(T_6 + \frac{1}{4\pi}\Big(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Big)\Big) \\ -c^2\varpi\varphi^*\Big(T_8 + \frac{1}{4\pi}\Big(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi ct}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi ct}{2k+1}\Big)\Big)\Big)\end{aligned} \tag{5.3}$$

for all values of n.

Conclusively, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

VI. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

To illustrate the analysis presented in this work, orthotropic rectangular plate is taken to be of length $L_y = 0.923m$, breadth $L_x = 0.432m$ the load velocity c=0.8123 m/s and $s = 0.4m$.The results are presented on the various graphs below for the simply supported boundary conditions.

6.1 Graphs for Clamped - Clamped End Conditions

Figures 6.1 and 6.2 display the effect of foundation modulus (Ko) on the deflection profile of clamped-clamped orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of foundation modulus (Ko) increases.

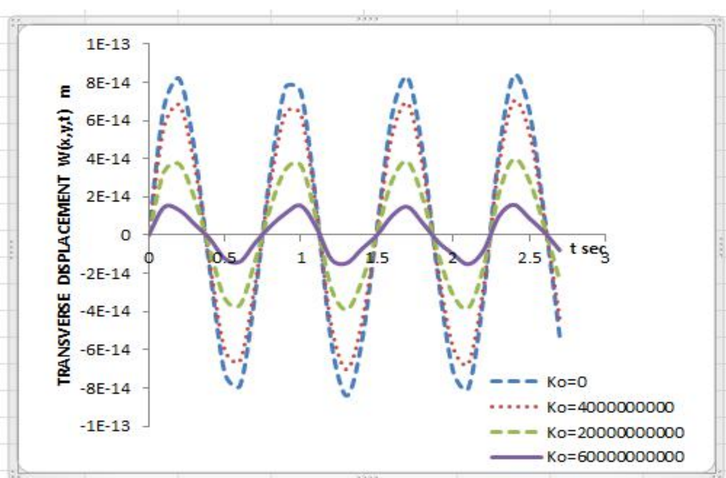

Fig.6.1: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Ko and Traversed by Moving Force

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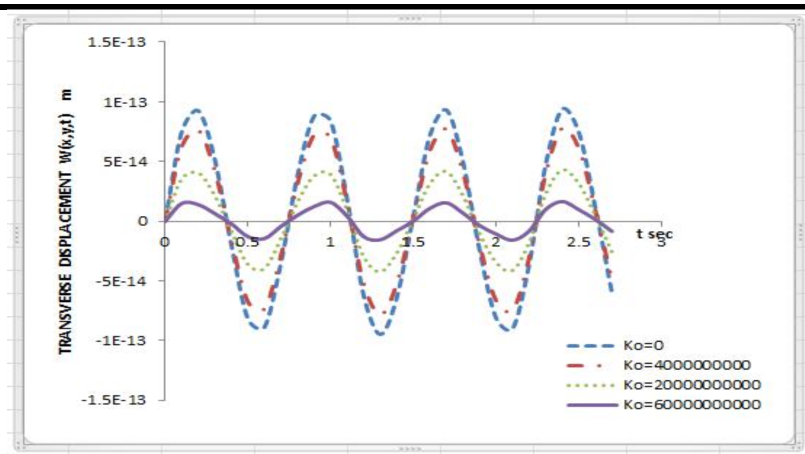

Fig.6.2: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Ko and Traversed by Moving Mass

Figures 6.3 and 6.4 display the effect of shear modulus (Go) on the deflection profile of clamped-clamped orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of shear modulus (Go) increases.

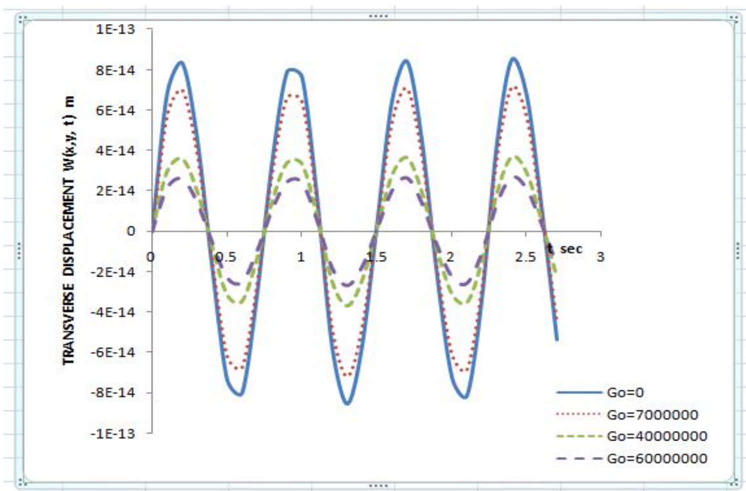

Fig.6.3: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Go and Traversed by Moving Force

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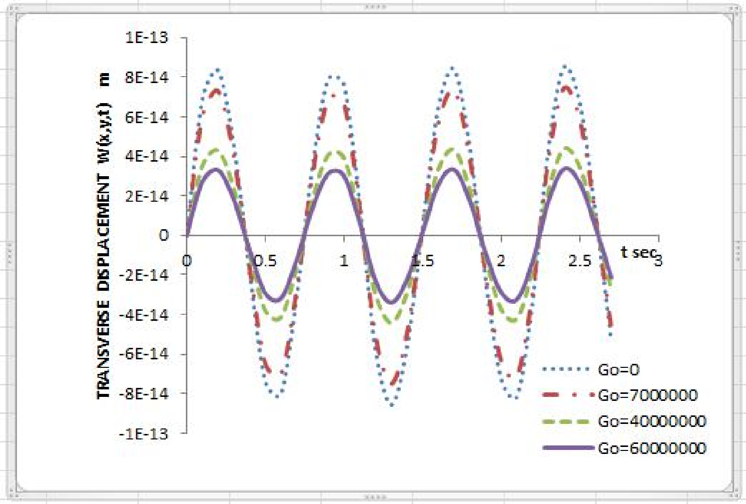

Fig.6.4: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Go and Traversed by Moving Mass

Figures 6.5 and 6.6 display the effect of flexural rigidity of the plate along x-axis (Dx) on the deflection profile of Clamped-clamped orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity (Dx) increases.

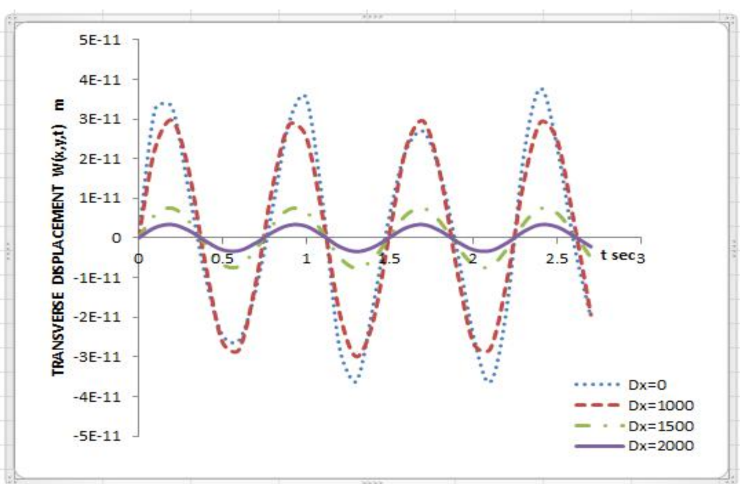

Fig.6.5: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Dx and Traversed by Moving Force

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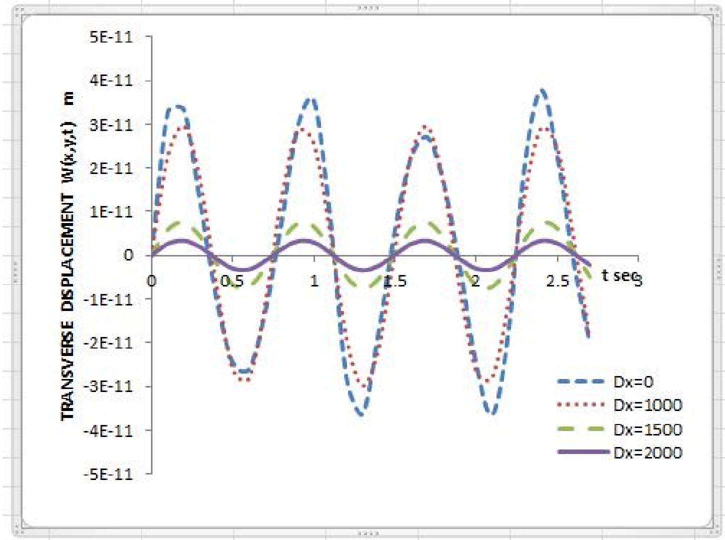

Fig.6.6: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Dx and Traversed by Moving Mass

Figures 6.7 and 6.8 display the effect of flexural rigidity of the plate along y-axis (Dy) on the deflection profile of Clamped-clamped orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity (Dy) increases.

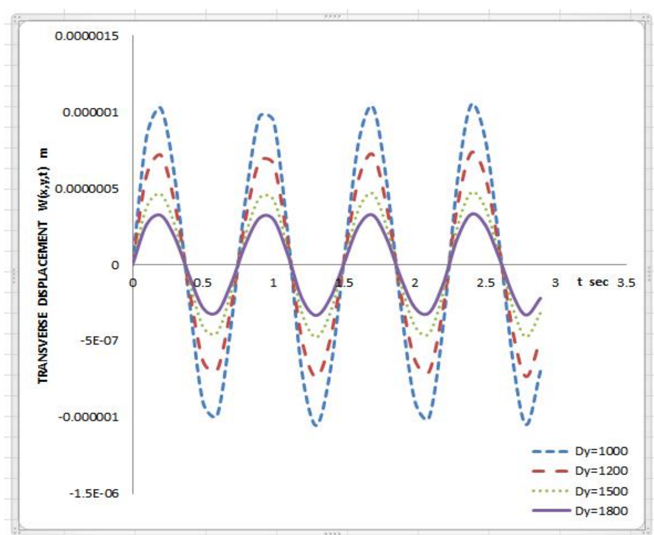

Fig.6.7: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Dy and Traversed by Moving Force

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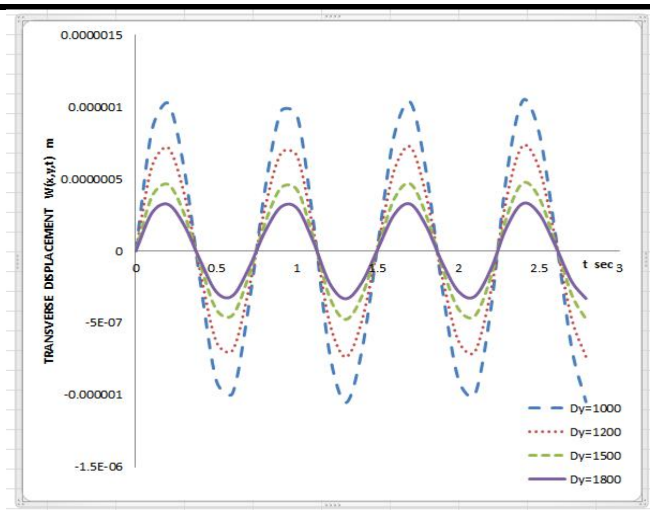

Fig.6.8: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Dy and Traversed by Moving Mass

Figures 6.9 and 6.10 display the effect of rotatory inertia (Ro) on the deflection profile of Clamped-clamped orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of rotatory inertia(Ro) increases.

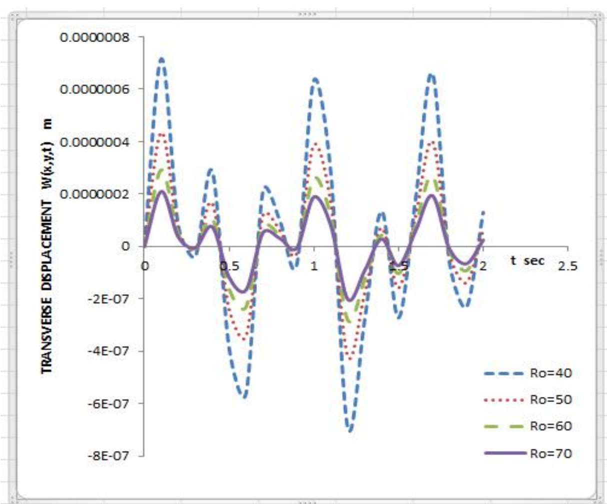

Fig.6.9: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Ro and Traversed by Moving Force

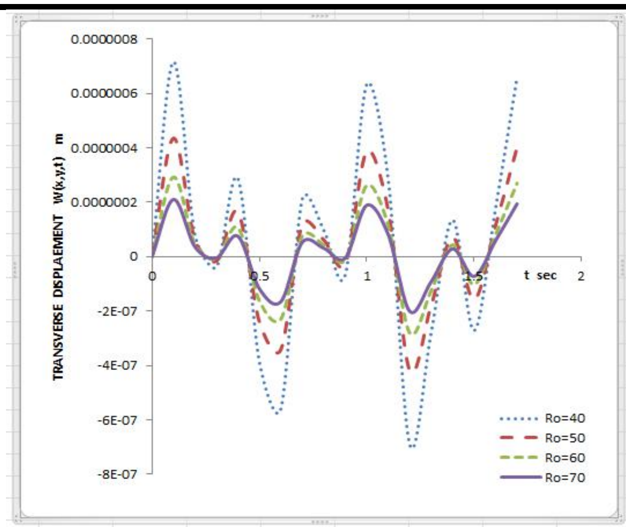

Fig.6.10: Displacement Profile of Clamped-clamped Orthotropic Rectangular Plate with Varying Ro and Traversed by Moving Mass

VII. CONCLUSION

In this research work, the problem of dynamic behavior of clamped orthotropic rectangular plates resting on bi-parametric foundation has been studied. The closed form solutions of the fourth order partial differential equations with variable and singular coefficients governing the orthotropic rectangular plates is obtained for both cases of moving force and moving mass using a method of Shadnam et al and another solution technique that is based on the separation of variables which was used to remove the singularity in the governing fourth order partial differential equation and thereby reducing it to a sequence of coupled second order differential equations. The modified Struble's asymptotic technique and Laplace transformation techniques are then employed to obtain the analytical solution to the two-dimensional dynamical problem.

The solutions are then analyzed. The analyses show that, for the same natural frequency and the critical speed for the moving mass problem is smaller than that of the moving force problem. Resonance is reached earlier in the moving mass system than in the moving force problem. That is to say the moving force solution is not an upper bound for the accurate solution of the moving mass problem. The results in plotted curves show that as the rotatory inertia correction factor (Ro) increases, the amplitudes of plates decrease for both cases of moving force and moving mass problems. The flexural rigidities along both the x-axis (Dx) and y-axis (Dy) increase, the amplitudes of plates decrease for both cases of moving force and moving mass problems. As the shear modulus (Go) and foundation modulus (Ko) increase, the amplitudes of plates decrease for both cases of moving force and moving mass problems.

It is shown further from the results that for fixed values of rotatory inertia correction factor, flexural rigidities along both x-axis and y-axis, shear modulus and foundation modulus, the amplitude for the moving mass problem is greater than that of the moving force problem which implies that resonance is reached earlier in moving mass problem than in moving force problem of clamped orthotropic rectangular plates resting on bi-parametric foundation.

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